

THREE DIMENSIONAL FIELD THEORIES FROM INFINITE DIMENSIONAL LIE ALGEBRAS

R.E.C. Perret

*Department of Physics
University of Southern California
Los Angeles, CA 90089-0484*

Submitted to: *Nuclear Physics B*

Abstract:

A procedure for constructing topological actions from centrally extended Lie groups is introduced. For a Kac-Moody group, this produces three dimensional Chern-Simons theory, while for the Virasoro group the result is a new three dimensional topological field theory whose physical states satisfy the Virasoro Ward identity. This topological field theory is shown to be a first order formulation of two dimensional induced gravity in the chiral gauge. The extension to W_3 -gravity is discussed.

USC-92/018

September 1992

1. Introduction

In recent years, numerous connections have been discovered between conformal field theories, integrable models, quantum groups and knot theory. A particularly illuminating framework for discussing these relationships is Chern-Simons theory in three dimensions [1]. Chern-Simons theory provides an intrinsically three dimensional description of many knot invariants and elucidates the relationship between knot theory and two dimensional statistical mechanics [2].

The starting point of this development was the realization by Witten that there is an intimate connection between three dimensional Chern-Simons theory and two dimensional current algebra [1]. The space of conformal blocks of the WZW model coincides with the space of physical states in the canonical quantization of Chern-Simons theory [1,3]. This three dimensional interpretation of the conformal blocks makes manifest some surprising symmetry of their braiding matrices [2]. It also gives, in the form of Wilson lines, an explicit representation of the Verlinde operators [4], which play an important role in the study of the modular properties of the conformal blocks [1].

In this paper I will show that the relationship between Chern-Simons theory and the WZW model can be understood directly from the underlying chiral algebra, the affine Kac-Moody algebra for a compact Lie group G . As is well known, one can express a character of a Lie algebra as a coherent state path integral [5,6,7]. The action in this path integral is the geometric action on the coadjoint orbit of the corresponding group—in the case of an affine Kac-Moody algebra, the action of the chiral WZW model [7,8,9]. This action does not define a consistent quantum field theory in its own right; rather, the corresponding (formally defined) partition function coincides with the chiral building blocks of the partition function of the full conformal field theory [9].

In Section 2, I formally consider the trace of the path ordered exponential of a two dimensional gauge field for a Kac-Moody group. Constructing a geometric action in exactly the same way as in the case of a character, I arrive at a gauged chiral WZW model coupled to a three dimensional Chern-Simons theory. The third component of the three dimensional gauge field is associated with the central extension of the algebra. The symmetry between all three components is the main nontrivial feature of the construction, and points to some hidden symmetry of the Kac-Moody algebra.

It is an interesting question to what extent this construction can be applied to other chiral algebras. If the chiral algebra of a conformal field theory can be related to a topological field theory in which Wilson like observables are defined, one would expect the latter to be related to the conformal blocks. Also, the expectation values of such observables might give rise to a nontrivial knot invariant.

In Section 3, I carry out the analogous construction for the Virasoro algebra. As in the Kac-Moody case there is indeed a nontrivial three dimensional symmetry, which is discussed in Section 4. In Section 5, I show that the geometric three dimensional action can be interpreted as a first order formulation for two dimensional chiral induced gravity, and I comment on the generalization to W -algebras. Although the nonlinearities present in W -algebras forbid a direct application of the path integral method, the action obtained in Section 3 has natural generalizations which are related to classical W -gravities. There does not seem to be any three dimensional symmetry in the W -case, but these actions do lead to some interesting results. In particular, they define a set of W -curvatures, from which natural geometric transformation rules for the W -gauge fields can be deduced. Also, they provide a convenient framework for the explicit construction of W -algebras, as discussed in [10].

2. Kac-Moody algebras and gauged chiral WZW models

In this section I introduce a path integral method for constructing topological actions using an affine Kac-Moody algebra as an example. I recover the well known relationship between Chern-Simons theory and chiral blocks of the WZW model. The method closely parallels the construction of the geometric action on the coadjoint orbits of a Kac-Moody group [7,8], which has been described in detail in [9].

Let G be a compact, connected, simply connected Lie group with Lie algebra \mathfrak{g} , and let \widehat{LG} denote the (universal) central extension of the loop group LG . The Lie algebra $\widehat{\mathfrak{lg}}$ of \widehat{LG} is defined by the following commutator

$$[(u_1, m_1), (u_2, m_2)] = ([u_1, u_2], \frac{i}{2\pi} \text{tr} \int dx u_2 \partial_x u_1). \quad (2.1)$$

Here $(u(x), m)$, with $u : S^1 \mapsto \mathfrak{g}$ and $m \in \mathbb{R}$, denotes an element of $\widehat{\mathfrak{lg}}$.

Consider a gauge potential \hat{A} for $\widehat{\mathfrak{lg}}$ defined on a two dimensional surface S , i.e. an $\widehat{\mathfrak{lg}}$ valued one-form $\hat{A} = (A(x), a)$ on S . Let γ be a contractible loop in S and Λ an irreducible unitary representation of LG with highest weight λ at level k , and consider the Wilson loop $\text{tr}_\Lambda \text{Pe}^{\oint_\gamma \hat{A}} q^{\mathcal{L}_0}$, where $q \equiv e^{i\tau}$ and $\mathcal{L}_0 \equiv i(\partial_x + A_x)$ is a covariantized rotation generator. A_x is a field on S with values in the loop algebra which under left translations $g \mapsto h^{-1}g$ transforms in the coadjoint representation of the Kac-Moody algebra

$$A_x \mapsto h^{-1}A_x h + h^{-1}\partial_x h. \quad (2.2)$$

I will show that A_x can be related to \hat{A} by a constraint which is consistent with this transformation property.

In analogy with the path integral representation of a character (see e.g. [6]), one can represent the Wilson loop as a coherent state path integral over the Kac-Moody group, whose elements are denoted by \hat{g}

$$\text{tr}_\Lambda \text{Pe}^{\oint_\gamma \hat{A}} q^{\mathcal{L}_0} = \int [d\hat{g}] e^{S(\hat{g}, \hat{A})} = \int [d\hat{g}] e^{\oint_\gamma \langle \hat{g}^{-1} d\hat{g} + \hat{g}^{-1}(\hat{A} + i\tau \mathcal{L}_0 dt) \hat{g} \rangle}. \quad (2.3)$$

Here, $\langle \hat{u} \rangle$ denotes the expectation value of $(u(x), m) \in \widehat{\mathfrak{lg}}$ in the highest weight state

$$\langle \hat{u} \rangle = \frac{1}{2\pi} \text{tr} \int dx \lambda u - km. \quad (2.4)$$

Explicitly, the action in (2.3) is

$$\begin{aligned} S(\hat{g}, \hat{A}) = & \frac{i}{2\pi} \text{tr} \int dx dt (\lambda g^{-1}(\bar{\partial} + \bar{A})g + \frac{ik}{2} g^{-1} \partial_x (\bar{\partial} + 2\bar{A})g) \\ & + \frac{ik}{12\pi} \text{tr} \int_V (g^{-1} dg)^3 + \frac{ik}{4\pi} \text{tr} \int dx dt (-\tau A_x^2) - k \oint_\gamma a, \end{aligned} \quad (2.5)$$

where $\bar{\partial} \equiv \partial_t - \tau \partial_x$ and $\bar{A} \equiv A_t - \tau A_x$ is the chiral component of a two dimensional gauge field minimally coupled to the Noether current corresponding to the global invariance under left translations of the ungauged action. The two dimensional integral is over a torus $T = \gamma \times S^1$ and $V = \Sigma \times S^1$ is the volume enclosed by T .

If \hat{A} is taken to be a constant element of the Cartan subalgebra of $\widehat{\mathfrak{lg}}$, $\hat{A} = (H dt, 0)$, with H in the Cartan subalgebra of \mathfrak{g} , and A_x is set to 0, the Wilson loop reduces to a character of $\widehat{\mathfrak{lg}}$, and the action becomes that of a (twisted) chiral WZW model. In that case, the path integral can be evaluated exactly and yields the Weyl-Kac character formula [9].

For general \hat{A} and A_x it is not clear how to interpret the last term in (2.5). As I show now, for a subclass of these fields it can be given an attractive interpretation in terms of three dimensional field theory.

The field strength $\hat{F} \equiv d\hat{A} + \frac{1}{2}[\hat{A}, \hat{A}]$ can be written as

$$\hat{F} = (F(x), f) = (dA + A^2, da - \frac{i}{4\pi} \text{tr} \int dx A \partial_x A) \quad (2.6)$$

and transforms in the adjoint representation

$$(F(x), f) \mapsto (h^{-1} F h, f - \frac{i}{2\pi} \text{tr} \int dx \partial_x h h^{-1} F). \quad (2.7)$$

The transformation rule of the central part f of the field strength implies that it is consistent with gauge invariance to impose the constraint

$$f = -\frac{i}{2\pi} \text{tr} \int dx A_x F, \quad (2.8)$$

where A_x is the field introduced above covariantizing ∂_x . Combining (2.6) and (2.8) one has then

$$da = -\frac{i}{4\pi} \text{tr} \int dx (2A_x F - A \partial_x A). \quad (2.9)$$

Given any $A(x)$ and A_x , this constraint can be solved at least locally for a . On the other hand, it is clear that not every \hat{A} admits a solution for A_x . In the following, I will restrict to the subclass \mathcal{A} of connections \hat{A} for which there exists an A_x which solves (2.9).

The loop group part $A(x)$ of \hat{A} defines two components of a three dimensional gauge potential for the finite dimensional group G , and A_x can be viewed as a third component for this gauge field. Thus, there is a correspondence between \mathcal{A} , the subclass of those two dimensional connections for the Kac-Moody group that admit a solution of (2.9), and the three dimensional connections for the finite dimensional group. The last term in (2.5),

$$\oint_{\gamma} a = \int_{\Sigma} da = -\frac{i}{4\pi} \text{tr} \int_{\Sigma} \int dx (2A_x F - A \partial_x A), \quad (2.10)$$

can be expressed as a three dimensional action for the three dimensional gauge field. Notice that the arbitrariness in A_x , given an $\hat{A} \in \mathcal{A}$, corresponds to a gauge invariance of this action, so that there is in fact a one-to-one correspondence between gauge equivalence classes in \mathcal{A} and gauge equivalence classes of three dimensional connections.

Given the completely asymmetric way in which A_x and the remaining two components have been introduced, one would not expect any three dimensional symmetry between these components. Surprisingly, the three components of A enter (2.10) in a completely symmetric way. In fact, (2.10) is, up to a boundary term, exactly a Chern-Simons theory on the manifold $\Sigma \times S^1$.

Thus, for $\hat{A} \in \mathcal{A}$, the action in the path integral expression (2.3) for the Wilson line is a gauged chiral WZW model including a chiral Chern-Simons term

$$S(\hat{g}, \hat{A}) = S_{\lambda}(g, \bar{A}) + S(A), \quad (2.11)$$

where

$$\begin{aligned} S_{\lambda}(g, \bar{A}) \equiv & \frac{1}{2\pi} \text{tr} \int dx dt (\lambda g^{-1} (\bar{\partial} + \bar{A})g + \frac{ik}{2} g^{-1} \partial_x (\bar{\partial} + 2\bar{A})g) \\ & + \frac{ik}{12\pi} \text{tr} \int_V (g^{-1} dg)^3 \end{aligned} \quad (2.12)$$

and

$$S(A) \equiv \frac{ik}{4\pi} \text{tr} \int dx dt \bar{A} A_x - \frac{ik}{4\pi} \text{tr} \int_V (AdA + \frac{2}{3} A^3). \quad (2.13)$$

By construction, this action should be invariant under left translations $g \mapsto h^{-1}g$, $A \mapsto h^{-1}Ah + h^{-1}dh$. Indeed, both (2.12) and (2.13) are invariant up to a chiral WZW model (i.e. an action of the form (2.12)) with $\lambda = 0$, with opposite signs. This is easily checked using the chiral version of the Polyakov-Wiegmann formula

$$S_{\lambda}(hg) = S_{\lambda}(g) + S_{g\lambda g^{-1}}(h) + \frac{ik}{2\pi} \text{tr} \int dx dt \partial_x g g^{-1} h^{-1} \bar{\partial} h, \quad (2.14)$$

where

$$S_{\lambda}(g) \equiv \frac{1}{2\pi} \text{tr} \int dx dt (\lambda g^{-1} \bar{\partial} g + \frac{ik}{2} g^{-1} \partial_x \bar{\partial} g) + \frac{ik}{12\pi} \text{tr} \int_V (g^{-1} dg)^3. \quad (2.15)$$

For comparison with the Virasoro case, which will be discussed in the next section, let me briefly summarize some points which show that the action (2.11) combines the main features of the relationship between Chern-Simons theory and the WZW model in an interesting way.

The equations of motion corresponding to (2.11) follow from

$$\begin{aligned} \delta S(g, A) = & \frac{1}{2\pi} \text{tr} \int dx dt (-\bar{D}(g\lambda g^{-1} + ik \partial_x g g^{-1}) - ik \partial_x \bar{A}) \delta g g^{-1} \\ & + \frac{1}{2\pi} \text{tr} \int dx dt (g\lambda g^{-1} + ik \partial_x g g^{-1} + ik A_x) \delta \bar{A} \\ & - \frac{ik}{2\pi} \text{tr} \int_V F \delta A. \end{aligned} \quad (2.16)$$

Choosing a radial time coordinate, perpendicular to the angular coordinate t and the loop group coordinate x , and performing canonical quantization on the torus T , the physical states of the pure Chern-Simons theory are characterized by the Ward identity [1]

$$F|_T = \bar{D}A_x - \partial_x \bar{A} = 0, \quad (2.17)$$

which is the projection of the three dimensional equation of motion on T . Here, A_x and \bar{A} are canonically conjugate variables

$$[\bar{A}, A_x] = \frac{2\pi}{ik}. \quad (2.18)$$

It is then easily seen, using the equations of motion in (2.16) and the gauge invariance of the measure, that (2.17) is exactly the anomalous Ward identity satisfied by the functionals

$$\psi_\lambda(\bar{A}) \equiv \int [dg] e^{S_\lambda(g, \bar{A})}. \quad (2.19)$$

Indeed, it is well known that the $\psi_\lambda(\bar{A})$ form a basis for the Hilbert space \mathcal{H} of physical states for Chern-Simons theory on $T \times \mathbb{R}$ [3]. As noted above, for \bar{A} a constant element of the Cartan subalgebra of \mathfrak{g} , they reduce to Weyl-Kac characters.

3. The Virasoro algebra

The method presented in the previous section can be applied to other centrally extended Lie algebras. Here, I will consider the case of the Virasoro algebra. Virasoro conformal blocks have been related to three dimensional topological field theory by H. Verlinde, who showed that in a specific, non gauge invariant polarization, the physical state condition in $SL(2, \mathbb{R})$ Chern-Simons theory leads to the Virasoro Ward identity [11]. However, to carry out quantization in such a non gauge invariant polarization seems problematic.

I will now construct an alternative topological action which can be related to Virasoro conformal blocks by following the procedure of the previous section. The construction of this action again parallels that of the geometric action on the coadjoint orbits [7,8]. I will denote elements of the Virasoro algebra Vir by $(g(x), n)$ and elements of the dual algebra Vir^* by $(b(x), c)$. Thus $g(x)\partial/\partial x$ is a vector field, while $b(x)dx^2$ is a quadratic differential on the circle S^1 . The algebra is defined by the commutator

$$[(g_1(x), n_1), (g_2(x), n_2)] = (g_1 g_2' - g_1' g_2, \frac{1}{48\pi} \int dx (g_1''' g_2 - g_1 g_2''')), \quad (3.1)$$

where the primes denote derivatives with respect to x , and the pairing between Vir and Vir^* is

$$\langle (g(x), n), (b(x), c) \rangle = \int dx g(x) b(x) + cn. \quad (3.2)$$

The corresponding Lie group is the central extension of the group of diffeomorphisms of the circle. It acts on Vir^* by the coadjoint action

$$\text{ad}_G^*(b(x), c) = (b(G(x))G'^2 - \frac{c}{24\pi} S(G), c), \quad (3.3)$$

where $G : x \mapsto G(x) \in \text{diff} S^1$ and $S(G)$ denotes the Schwarzian derivative of G .

A character of an irreducible unitary representation can be expressed as a coherent state path integral by going through the same steps which lead to (2.3)

$$\text{tr} q^{L_0} = \int [dG] e^{\oint d^{-1} \Omega + \int dt \langle \text{ad}_{G^{-1}}(i\tau L_0) \rangle}, \quad (3.4)$$

where Ω is the symplectic form on the coadjoint orbit [12,7,8]

$$\Omega(G) = \langle (\frac{dG}{G'})^2 \rangle, \quad (3.5)$$

and $\langle X \rangle$ denotes the pairing of $X \in g$ with the coadjoint vector (b_0, c) . For simplicity, I am choosing $b(x) = b_0$ to be a constant differential. The explicit form of the action in (3.4) is

$$S = \int dx dt (b_0 G' \bar{\partial} G + \frac{c}{48\pi} \frac{\bar{\partial} G''}{G'}). \quad (3.6)$$

For $\tau = 0$, this action is equivalent to the one derived in [7,8]. As previously, I introduce a gauge field $\hat{A} = (\mu(x), a)$. Here $\mu(x)$, which is a gauge potential for $\text{diff} S^1$, transforms as a Beltrami differential

$$\mu \mapsto \mu \circ H H'^{-1} - dH H'^{-1} \quad (3.7)$$

under the substitution $G \mapsto G \circ H$. The corresponding field strength $\hat{F} \equiv d\hat{A} + \frac{1}{2}[\hat{A}, \hat{A}]$ can be written as

$$\hat{F} = (F(x), f) = (d\mu + \mu\mu', da - \frac{1}{48\pi} \int dx \mu\mu''') \quad (3.8)$$

and transforms in the adjoint representation

$$\begin{aligned} \hat{F} \mapsto \text{ad}_H \hat{F} &= (F \circ H H'^{-1}, f - \frac{1}{24\pi} \int dy S(H^{-1}) F) \\ &= (F \circ H H'^{-1}, f + \frac{1}{24\pi} \int dx S(H) F \circ H H'^{-1}). \end{aligned} \quad (3.9)$$

The transformation law of f implies that it is consistent to impose a constraint

$$f = \frac{1}{24\pi} \int dx T F, \quad (3.10)$$

where T transforms in the coadjoint representation

$$T \mapsto T \circ H H'^2 + S(H). \quad (3.11)$$

Eq. (3.10) is the analogue of eq. (2.8). In the following, I restrict to such \hat{A} that (3.10) has a solution for T . The path integral expression for a Wilson loop for \hat{A} is analogous to (3.4)

$$\text{tr} P e^{\oint \hat{A}} = \int [dG] e^{\oint (d^{-1}\Omega + \langle \text{ad}_{G^{-1}} \hat{A} \rangle)}. \quad (3.12)$$

The second term in the action

$$\oint \langle \text{ad}_{G^{-1}} \hat{A} \rangle = \int dx dt \mu (b_0 G'^2 - \frac{c}{24\pi} S(G)) + c \oint a \quad (3.13)$$

contains, besides a coupling to the Noether current, again a term involving the central part of the gauge potential. Using the same trick as in the Kac-Moody case, I write this as a three dimensional action

$$c \oint a = c \int_{\Sigma} da = \frac{c}{48\pi} \int_{\Sigma} \int dx (2TF + \mu\mu'''). \quad (3.14)$$

The action in the path integral (3.12) can now be written explicitly as the sum of a two dimensional and a three dimensional part

$$S(G, \hat{\mu}) = S_2(G, \mu_t) + S_3(\mu, T), \quad (3.15)$$

with

$$S_2(G, \mu_t) = \int dx dt (b_0 G' (\dot{G} + \mu_t G') + \frac{c}{48\pi} (\frac{\dot{G}''}{G'} + 2\mu_t S(G)) \quad (3.16)$$

and

$$S_3(\mu, T) = \frac{c}{48\pi} \int_{\Sigma} \int dx (2TF + \mu\mu'''). \quad (3.17)$$

Again, it can be shown that the total action is gauge invariant, although this is less straightforward than in the Kac-Moody case. The analogue of the Polyakov-Wiegmann formula is

$$\begin{aligned} \int dx dt \frac{(G \circ H)''}{(G \circ H)'} &= \int dx dt \frac{\dot{G}''}{G'} \circ HH' + \int dx dt \frac{\dot{H}''}{H'} \\ &\quad - 2 \int dx dt S(G) \circ H \dot{H} H', \end{aligned} \quad (3.18)$$

from which it follows that

$$S_2(G \circ H, \mu_t \circ HH'^{-1} - \dot{H} H'^{-1}) = S_2(G, \mu_t) + S_2(H, \mu_t \circ HH'^{-1} - \dot{H} H'^{-1})|_{b_0=0}. \quad (3.19)$$

As in the Kac-Moody case, the three dimensional action is invariant up to a boundary term. This boundary term is precisely the two dimensional action without the b_0 term

$$\begin{aligned} S_3(\mu \circ HH'^{-1} - dHH'^{-1}, T \circ HH'^2 + S(H)) = \\ S_3(\mu, T) - S_2(H, \mu_t \circ HH'^{-1} - \dot{H} H'^{-1})|_{b_0=0}, \end{aligned} \quad (3.20)$$

and cancels the second term in (3.19).

The equations of motion corresponding to (3.15) follow from

$$\begin{aligned} \delta S &= \int dx dt (-\nabla_t(b_0 G'^2 - \frac{c}{24\pi} S(G)) + \frac{c}{24\pi} \mu_t''') \frac{\delta G}{G'} \\ &\quad + \int dx dt (b_0 G'^2 - \frac{c}{24\pi} S(G) + \frac{c}{24\pi} T) \delta \mu_t \\ &\quad + \frac{c}{24\pi} \int_{\Sigma} \int dx (\nabla T + \mu''') \delta \mu. \end{aligned} \quad (3.21)$$

Here, $\nabla\phi$ denotes the covariant derivative

$$\nabla\phi \equiv d\phi + \mu\phi' + s\mu'\phi \quad (3.22)$$

of a field with spin s , i.e., the homogeneous part of whose transformation rule is $\phi \mapsto \phi \circ H(H')^s$. Notice that (3.21) is completely analogous to the corresponding equation in the Kac-Moody case (2.16).

Choosing the radial coordinate of the disk Σ as time coordinate, and performing canonical quantization, one finds that T and μ_t are canonically conjugate

$$[\mu_t, T] = \frac{24\pi}{c}. \quad (3.23)$$

The physical state condition following from (3.21)

$$(\nabla T + \mu''')|_{\Sigma} = 0 \quad (3.24)$$

is the Virasoro Ward identity [11]. This is the defining equation for conformal blocks of the partition function for conformal field theories whose maximal chiral algebra is the Virasoro algebra. The partition function of the two dimensional action

$$\psi(\mu_t) \equiv \int [dG] e^{S_2(G, \mu_t)} \quad (3.25)$$

is easily seen, using (3.21), to solve the Ward identity (3.24). Thus, it defines a physical state of the three dimensional theory.

4. Covariant form of the three dimensional action

In the previous section, I obtained a three dimensional topological field theory associated to the Virasoro algebra coupled to the chiral part of a two dimensional conformal field theory on the boundary. This two dimensional theory defines states in the Hilbert space associated to canonical quantization of the three dimensional theory.

At this point, the three dimensional action (3.17) looks rather asymmetric between the Beltrami differential μ and the field T . I will show now that there is in fact a remarkable symmetry between the two, and that recognizing this symmetry enables one to rewrite the action S_3 in a manifestly generally covariant form.

I restricted to those Virasoro connections which can be parametrized as

$$\hat{A} \equiv (\mu, d^{-1}(\frac{c}{48\pi} \int dx (2TF + \mu\mu'''))). \quad (4.1)$$

The only relevant property of T in this expression is its transformation rule (3.11), which says that T transforms as a stress tensor. It is well known that one can parametrize T in terms of a so called affine connection λ (see e.g. [13])

$$T \equiv -(\lambda' + \frac{1}{2}\lambda^2), \quad (4.2)$$

where λ transforms in the following way

$$\lambda \mapsto \lambda \circ HH' - \frac{H''}{H'}. \quad (4.3)$$

The affine connection λ can be used to define covariant derivatives with respect to x on objects that transform as tensors under $\text{diff} S^1$ (remember that μ defines covariant *exterior* derivatives, cf. (3.22)). Indeed, it is easily checked that for ϕ a tensor of spin s

$$D\phi \equiv \phi' + s\lambda\phi \quad (4.4)$$

is a tensor of spin $s + 1$.

From λ and μ , one can construct a three dimensional gauge field

$$A \equiv D\mu + \lambda dx. \quad (4.5)$$

Using (3.7) and (4.3), it is straightforward to verify that A transforms as

$$A \mapsto A - d \log H', \quad (4.6)$$

where d is now the exterior derivative in three dimensions, $d \equiv d|_{\Sigma} + dx \partial_x$. Moreover, up to a boundary term, the action (3.14) can be rewritten as an abelian Chern-Simons action

$$S_3(\mu, T) = \int_V AdA - \int dx dt \mu'_t \lambda. \quad (4.7)$$

Hence, there is a very remarkable symmetry in three dimensions between the Beltrami differential μ and the affine connection λ .

Obviously, the Wilson loops

$$\text{tr}_s P e^{\oint A} = P e^{s \oint A} \quad (4.8)$$

are gauge invariant observables of the three dimensional theory. Canonical quantization on a space Σ , pierced by a collection of such Wilson loops corresponding to spins s_i , leads to the following Ward identity

$$\nabla_t T + \mu_t''' + \frac{24\pi}{c} \sum_i (s_i \delta'(z - z_i) + \delta(z - z_i) \partial / \partial_{x_i}) = 0, \quad (4.9)$$

where $z = (x, t)$ and $z_i = (x_i, t_i)$ denote the positions of the punctures. This is indeed the correct Ward identity defining the Virasoro conformal blocks [11]. The states satisfying this Ward identity can be expressed as

$$\psi(\mu_t, z_i) \equiv \int [dG] \prod_i (G'(z_i))^{s_i} e^{S_2(G, \mu_t)}. \quad (4.10)$$

One would expect, by analogy with the WZW model, that these Wilson loops represent the Verlinde operators, and can be used to construct knot invariants.

5. Classical W-gravity

The equations of motion (3.21) of the Virasoro action are solved by the pure gauge configurations $\mu = -\frac{dH}{H}$, $T = S(H)$. Substituting these expressions in the three dimensional action (3.17) and using the transformation property (3.20), one sees that the action reduces to a pure boundary term which is exactly Polyakov's induced gravity in the chiral gauge [14]. In this sense, the three dimensional action can be interpreted as a first order formulation of Polyakov's gravity. This can also be seen more directly as follows. Upon applying the three dimensional equations of motion, the variation of the Virasoro action is a boundary term

$$\delta S = -\frac{c}{24\pi} \int dx dt T \delta \mu = \frac{c}{24\pi} \int dx dt \nabla^{-1} \mu''' \delta \mu. \quad (5.1)$$

This is the anomaly defining the induced action.

The induced action for W -gravity theories has a rather complicated structure, and in particular includes nonlocalities in the quantum case [15]. The three dimensional first order formulation of induced gravity, as discussed above, can be extended straightforwardly to first order formulations of classical W - and super W -gravities. A detailed construction of such extensions is given in [10]. In the following, I will discuss the example of W_3 -gravity. The action in this case is [10]

$$S = \int_{\Sigma} \int dx (2TF + 2W\nabla\nu + \mu\mu''' + \nu\nu^{(5)} + 10\nu\nu'''T - 6\nu\nu'T'' + 16\nu\nu'T^2). \quad (5.2)$$

Here, W is the spin 3 field and ν is a spin -2 one-form, the gauge field of the W -symmetry. The equations of motion are the classical W_3 Ward identities (see e.g. [15])

$$\nabla T + \mu''' + 3\nu'W + 2\nu W' = 0, \quad (5.3)$$

$$\nabla W + D^5\nu = 0, \quad (5.4)$$

where D^5 is the second Gelfand-Dickey operator, and the Maurer-Cartan equations

$$d\mu + \mu\mu' + 2\nu\nu''' - 3\nu'\nu'' + 16\nu\nu'T = 0, \quad (5.5)$$

$$\nabla\nu = 0. \quad (5.6)$$

Eqs. (5.5,6) define an algebra with field dependent structure constants. Together with eqs. (5.3,4) they form an integrable system, i.e., the integrability condition $d^2 = 0$ is identically satisfied.

In the standard way one derives the gauge transformation laws for the gauge potentials μ and ν

$$\delta\mu = \nabla\epsilon + 2\nu\eta''' - 2\nu''' \eta - 3\nu'\eta'' + 3\nu''\eta' + 16\nu T\eta' - 16\nu'T\eta, \quad (5.7)$$

$$\delta\nu = \nabla\eta + 2\nu\epsilon' - \nu'\epsilon. \quad (5.8)$$

Similarly, the Ward identities (5.3,4) determine, by contraction with the gauge parameters, the standard transformation laws of T and W

$$\delta T = -2\epsilon'T - \epsilon T' - \epsilon''' - 3\eta'W - 2\eta W', \quad (5.9)$$

$$\delta W = -3\epsilon'W - \epsilon W' - D^5\eta. \quad (5.10)$$

The variation of the action under these gauge transformations is

$$\delta S = -2 \int dx dt (\epsilon\mu''' + \eta(\nu^{(5)} - 16\nu'T^2 - 16\nu T T')). \quad (5.11)$$

The transformation laws (5.7–10) lead to the following transformation laws for the curvatures (5.3–6)

$$\begin{aligned} \delta\hat{F} = & -\epsilon\hat{F}' + \epsilon'\hat{F} - 2\eta\nabla\nu''' + 2\eta''' \nabla\nu + 3\eta'\nabla\nu'' - 3\eta''\nabla\nu' \\ & - 16\eta\nabla\nu'T + 16\eta'\nabla\nu T + 16\eta\nu'\hat{\nabla}T - 16\eta'\nu\hat{\nabla}T, \end{aligned} \quad (5.12)$$

$$\delta\nabla\nu = 2\epsilon'\nabla\nu - \epsilon\nabla\nu' - 2\eta\hat{F}' + \eta'\hat{F}, \quad (5.13)$$

$$\delta\hat{\nabla}T = -2\epsilon'\hat{\nabla}T - \epsilon(\hat{\nabla}T)' - 3\eta'\hat{\nabla}W - 2\eta(\hat{\nabla}W)', \quad (5.14)$$

$$\delta\hat{\nabla}W = -3\epsilon'\hat{\nabla}W - \epsilon\hat{\nabla}W', \quad (5.15)$$

where $\hat{\nabla}T$, $\hat{\nabla}W$, \hat{F} and $\nabla\nu$ are the left hand sides of (5.3–6), respectively. As in an ordinary Lie algebra, the curvatures transform homogeneously, so that the equations of motion, which state the vanishing of these curvatures, are gauge covariant. Hence, eqs. (5.9,10) are still valid in second order formalism, in which T and W become functions of μ and ν so that their transformation laws are no longer independent. The variation (5.11) is thus in fact the variation of the induced gravity action. All this is in agreement with the classical limit of the corresponding results obtained in [15] for quantum induced gravity.

6. Conclusion

I have presented in this paper three dimensional actions related to both linear and nonlinear chiral algebras of importance in conformal field theory, as well as a general procedure to construct such actions in the linear case. A simpler procedure to construct these actions which generalizes to nonlinear algebras is described in [10]. The equations of motion corresponding to these actions include the Ward identities of the algebra, as well as Maurer-Cartan equations for the gauge potentials. The action associated to the Virasoro algebra is a first order formulation of two dimensional induced gravity in the chiral gauge, while the action associated to the W_3 -algebra is a first order formulation of the classical limit of W_3 -gravity. The Virasoro and Kac-Moody actions have a surprising three dimensional symmetry, the latter being the action for Chern-Simons theory for the corresponding finite dimensional group. The Virasoro action can be written in the form of an abelian Chern-Simons theory, and admits the definition of Wilson loops. These Wilson loops create insertions in the conformal blocks, and might be of interest in knot theory.

Acknowledgements:

I would like to thank K. Pilch for suggesting many improvements to the manuscript and for illuminating discussions. This research was partially supported by the Department of Energy Contract #DE-FG03-91ER40168.

References

- [1] E. Witten, Commun. Math. Phys. **121** (1989) 351
- [2] E. Witten, Nucl. Phys. **B322** (1989) 629; Nucl. Phys. **B330** (1990) 285
- [3] M. Bos and V.P. Nair, Phys. Lett. **223B** (1989) 61; Int. J. Mod. Phys. **A5** (1990) 959
S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, Nucl. Phys. **B326** (1989) 108
- [4] E. Verlinde, Nucl. Phys. **B300** (1988) 360
- [5] A. Alekseev, L. Faddeev and S. Shatashvili, J. Geom. Phys. **1** (1989) 3
- [6] M. Stone, Nucl. Phys. **B314** (1989) 557
- [7] A. Alekseev and S. Shatashvili, Nucl. Phys. **B323** (1989) 719
- [8] P.B. Wiegmann, Nucl. Phys. **B323** (1989) 311
- [9] R.E.C. Perret, Nucl. Phys. **B356** (1991) 229
- [10] R.E.C. Perret, USC-92/016 (hep-th/9208068)
- [11] H. Verlinde, Nucl. Phys. **B337** (1990) 652

- [12] E. Witten, Commun. Math. Phys. **114** (1988) 1
- [13] M. Bauer, P. Di Francesco, C. Itzykson and J.-B. Zuber, Nucl. Phys. **B362** (1991) 515
- [14] A.M. Polyakov, Mod. Phys. Lett. **A2** (1987) 893
- [15] H. Ooguri, K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Commun. Math. Phys. **146** (1992) 616